Comparison of Spectral Method and Lattice Boltzmann Simulations of Two-Dimensional Hydrodynamics

D.O.Martínez, W.H.Matthaeus, S.Chen*, and D.C.Montgomery†
Bartol Research Institute
University of Delaware, Newark, DE 19716

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Abstract

We present numerical solutions of the two-dimensional Navier-Stokes equations by two methods; spectral and the novel Lattice Boltzmann Equation (LBE) scheme. Very good agreement is found for global quantities as well as energy spectra. The LBE scheme is, indeed, providing reasonably accurate solutions of the Navier-Stokes equations with an isothermal equation of state, in the nearly incompressible limit. Relaxation to a previously reported "sinh-Poisson" state is also observed for both runs.

KEY WORDS: Lattice Boltzmann method, hydrodynamics, spectral methods, nearly incompressible flows.

*Center for Nonlinear Studies and Theoretical Division, MS-B213, Los Alamos National Laboratory, Los Alamos, NM 87545
†Department of Physics and Astronomy, Dartmouth College, Hanover

1 Introduction

In recent years[1, 2, 3], lattice gas models have been developed for a number of fluid and fluid-like systems. Interest in the study of these methods derives from both theoretical and practical motivations. On the one hand, lattice gases provide a novel perspective on complex physical systems, differentiating the macroscopic physical effects that are observable from the simplified microscopic properties that ultimately are responsible for what is observed. From a computational perspective, lattice gases have shown promise as an alternative approach for solution of fluid-like partial differential equations, especially on parallel computers where the relative independence of the lattice nodes can be exploited for computational efficiency. Lattice gas models of the Cellular Automaton (CA) type have been developed for many systems, including hydrodynamics[1], magnetohydrodynamics[4] (MHD), multi-phase flows[5], and flows through porous media[6, 7]. Generally speaking CA models are plagued with noise, so that very large spatial grids must be used. Schemes to improve the situation are made difficult by complexity of collisions, difficulty in eliminating spurious modes, lack of Galilean invariance, and other problems. Nevertheless, some appealing results have been obtained using CA fluid models. Lattice Boltzmann (LBE) methods[8, 9] represent an improvement in terms of noise and have produced promising results in computations. Many of the previous demonstrations of lattice gas have shown clearly that complex physical fluid phenomena can be reproduced by these methods. Examples include wave propagation in hydrodynamics and MHD, vortex "streets" in viscous flows around obstacles, immiscible fluid effects, and others. However, most and perhaps all of these demonstrations have been confined to a qualitative verification of the physics, and have stopped short of showing that the lattice methods in fact provide an alternative method for quantitatively accurate solution of the fluid equations.

Since the introduction of LBE methods there has been recognition that the LBE framework provides opportunity to eliminate some, and perhaps all of the fundamental problems in the lattice gas approach. Two of these improvements are the use of the "single time relaxation approximation" (STRA) collisions[10, 11], and the ability to introduce corrections to the pressure that modify the equation of state and eliminate spurious modes[11, 12]. In addition, the LBE approach permits greater flexibility in implementations in terms of lattice structure and dynamical "rules".

Here we present a study of an improved LBE method in two dimensional (2D) hydrodynamics, demonstrating that for a simple nonlinear shear layer problem, the LBE method is accurate and effective. Related study of the performance of a 3D LBE scheme has been presented by Chen et al[13]. The problem we choose to address is a simple 2D periodic shear layer, perturbed by the addition of a low level of random "noise". This basic flow problem is of broad relevance to flow applications in geophysics, aerodynamics and space physics, and has been studied in laboratory situations and through numerical simulations. Although three dimensional effects are absent in this treatment of the nonlinear shear instability problem, and the 2D physical phenomena are well known, the simplicity and familiarity of this problem makes it a good starting place for accurate demonstration of the LBE method.

The physical effects we are interested in reproducing are: spectral transfer, energy and enstrophy decay, relaxation to the long time "maximum entropy state" at times shorter than viscous decay time. Although our primary purpose is to examine the incompressible behavior, the improved LBE scheme also is seen to accurately provide information about the "nearly incompressible" features of the dynamics, including waves and nearly incompressible pressure and density fluctuations.

2 Lattice Boltzmann Method

We adopt a numerical scheme appropriate to 2D hydrodynamics that is based upon the Lattice Boltzmann Equation, giving rise to the abbreviation "LBE" method. This approach to solution of fluid equations is based upon the kinetic equations associated with Cellular Automaton (CA) models for fluids[1,2,3]. In the CA formulation, the Boltzmann equation does not enter into numerical implementations, since it is constructed solely to demonstrate that certain averaged functions of the lattice dynamics approach solutions of the fluid equations. The LBE method arises from the suggestion[8] that a direct solution of these equations would provide an alternative numerical approach to computation, conceptually midway between the Boolean CA dynamics and the continuum fluid equations.

The several types of LBE models proposed thus far share with one another the advantage, relative to the underlying CA model, of significantly reduced noise. However, it has also been recognized that the LBE approach

allows for simplifications to improve numerical efficiency, as well as improvements to "cure" problems that arise in the underlying CA models. These refinements have substantially improved the prospect of useful LBE computations. The first major LBE refinement was the recognition that the "exact" LBE collision integral is unnecessarily complex and numerically inefficient[9]. The first idea for streamlining the evaluation of the collision integral was[9] to linearize the exact Boltzmann form. Evidently such a simplification preserves the tendency to approach the desired local equilibrium microscopic state, which is already known (from CA theory) to lead macroscopically to hydrodynamics. The only cost is a certain amount of departure of the distribution from what it would be in the CA case. However, since the departures from equilibrium are generally assumed to be small, this is not expected to produce discrepancies in the physical results. Expanding on the idea that the details of the collision operator need not correspond to the Boltzmann approximation to the exact CA rules, two groups nearly simultaneously offered the suggestion[10, 11, 12] that the exact collision operator can be, in effect, discarded, provided that one adopts a collision operator that leads, in a controllable fashion, to a desired local equilibrium state. By a "desired" equilibrium, we mean (1) one that depends only upon the local fluid variables, which themselves can be computed from the actual values of the local distribution at a point, (2) one that leads to the desired macroscopic equations (e.g., the Navier Stokes equation), and (3) one that admits whatever additional properties that are sought, such as simplicity or removal of nonphysical lattice effects. Recent work has shown that property (2) can be maintained rather easily, even when the collision operator departs significantly from the form taken in the Boltzmann treatment of the CA. In fact such departures are desirable from the point of view of several factors of type (3).

Chen et al[10] offered the first suggestion that one could use the single time relaxation, or STRA, collision operator for a MHD LBE method. Subsequently, a similar method[11] was described, and referred to as a "BGK" collision integral, in reference to the more elaborate collision treatment of Bhatnagar, Gross and Krook[14]. The essence of the suggestion for the LBE method is that the collision term $\Omega(f)$ be replaced by the well known classical single time relaxation approximation

$$\Omega(f) = -\frac{f - f^{eq}}{\tau}.$$

An appropriately chosen equilibrium distribution is denoted by f^{eq}, which

depends upon the local fluid variables, and a lattice relaxation time τ that controls the rate of approach to this equilibrium. Later, Qian et al[11] and Chen et al[12] described a STRA method for hydrodynamics that incorporates the form (1), but which also includes a reservoir of "stopped" particles that enter into the equilibrium distribution to prevent the particle distribution from "cooling" in regions of higher fluid speed. The latter problem had plagued earlier CA implementations of fluid models by giving rise to a velocity dependent pressure. An improper equation of state of this kind introduces nonphysical compressive effects [15, 16], including spurious oscillations, and incorrect pressure profiles in channel flows. These effects are completely eliminated, to all orders in the Mach number, by these "pressure corrected" LBE schemes[11, 12]. In contrast, multispeed CA models only partially correct the equation of state, by moving the velocity dependence of the pressure to higher order. Still further improvements to the method were realized when the stopped particle reservoir was parameterized in such a way[17] that the sound speed could be controlled, enabling higher Mach number flows, and in principle, shocks, to be computed with the STRA-LBE scheme.

In the subsequent sections, we present results obtained with an LBE scheme for 2D hydrodynamics, that incorporates a number of the above described features. We use a square lattice with eight moving particle directions plus stopped "particles"[11, 18]. In CA terminology, this "9-bit model" refers to a lattice dynamical system in which particles stream from nodes on the lattice to the nearest neighbor nodes at fixed speeds, experiencing collisions at each node, which modify the particle state, and on average drive the particle distribution toward equilibrium. Nearest neighbor nodes relative to a node at $\mathbf{x}$ are located at the face-centers $\mathbf{x} + \mathbf{c}_a^I$, for $a = 1, 2, 3, 4$, with $\mathbf{c}_a^I \equiv (\cos{(a-1)\pi/2}, \sin{(a-1)\pi/2})$, and the vertices of the square centered about $\mathbf{x}$, i.e., $\mathbf{x} + \mathbf{c}_a^{II}$, for $a = 1, 2, 3, 4$, with $\mathbf{c}_a^{II} \equiv \sqrt{2}(\cos{(a-1/2)\pi/2}, \sin{(a-1/2)\pi/2})$. To move to the appropriate node during the streaming step, a particle in state I_a moves with velocity $\mathbf{c}_a^I$ while particles in the state II_a move with velocity $\mathbf{c}_a^{II}$. In "lattice units", the lattice side can be taken to be $\delta x = 1$ and the lattice streaming time $\delta t = 1$, so that type I particles have unit speed and type II particles have speed $\sqrt{2}$.

Turning to an LBE treatment of the dynamics, we denote the moving particle distribution function by f_a^k for $k = I$ or II and $a = 1, 2, 3, 4$, while the component of the particle distribution referring to the stopped particles

(which do not stream) is designated by f_0. Adopting a single time collision operator, along with the foregoing streaming rules, we arrive at a kinetic equation

$$f_a^k(\mathbf{x}+\mathbf{c}_a^k, T+1) - f_a^k(\mathbf{x}, T) = -\frac{f_a^k - f_a^{k(eq)}}{\tau} \qquad (1)$$

where $k = I$ or II.

Our LBE dynamical system is completed by choosing the equilibrium distribution[11],

$$\begin{aligned} f_0 &= \frac{4}{9}\rho[1 - \frac{3}{2}u^2] \\ f_a^I &= \frac{\rho}{9}[1 + 3\mathbf{c}_a^I \cdot \mathbf{u} + \frac{9}{2}(\mathbf{c}_a^I \cdot \mathbf{u})^2 - \frac{3}{2}u^2] \\ f_a^{II} &= \frac{\rho}{36}[1 + 3\mathbf{c}_a^{II} \cdot \mathbf{u} + \frac{9}{2}(\mathbf{c}_a^{II} \cdot \mathbf{u})^2 - \frac{3}{2}u^2] \end{aligned} \qquad (2)$$

where the mass density ρ and fluid velocity $\mathbf{u}$ are defined by

$$\rho = f_0 + \sum_{k,a} f_a^k \qquad (3)$$

and

$$\rho\mathbf{u} = \sum_{k,a} \mathbf{c}_a^k f_a^k. \qquad (4)$$

Several fundamental properties of this LBE scheme can be readily demonstrated, based upon the choice of equilibrium and the kinetic equation (1). It is straightforward to show that the pressure $p = C_s^2\rho$, where $C_s = 1/\sqrt{3}$ is the sound speed. This is a "pressure corrected" LBE scheme with an exact isothermal equation of state. Next, considering moments of the kinetic equation, expanded according to a multiple scale Chapman-Enskog procedure, we find that the long wavelength low frequency behavior corresponds, at leading order, to a fluid equation for the velocity field. In addition, if one invokes a low Mach number ordering, which allows an approach to incompressible behavior, one obtains, in first approximation, the incompressible Navier Stokes equations,

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_0}\nabla p^\infty + \nu\nabla^2\mathbf{v}, \qquad (5)$$

where p^∞ is the incompressible pressure, and ρ_0 is the conserved initial mean density. Likewise, the small time dependent density variations obey

a continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{6}$$

At next order in the Chapman Enskog expansion, making use of the STRA collision operator with constant relaxation time scale τ, we find that the viscosity is $\nu = (2\tau - 1)/6$ ($\tau > 0.5$), which has the computationally desirable property of being independent of the density. Further discussion of the approach to incompressibility is given in section 6, and some additional remarks concerning the viscosity and the physical interpretation of τ are given in the Appendix.

3 Shear Layer Simulations: Spectral and Lattice Boltzmann

The idealized shear layer consists of uniform velocity reversing sign in a very narrow region. That corresponds to a vorticity $\omega = (\nabla \times \mathbf{v})_z$ different from zero only in the region of the sheared flow, and it is, in the ideal situation, a delta function. Therefore, we generated our initial conditions, in a simulation domain that is a square box of side 2π, with a spectral representation of delta functions at $y = \pi/2$ and $y = 3\pi/2$ (with opposite signs) for the vorticity, truncated to include the appropriate Fourier amplitudes with wavevectors $k = 1$ through 8. This configuration is steady in the absence of viscosity and, although the simulations are viscous, we add a perturbation to trigger the nonlinear terms of the Navier Stokes equation. To this end the velocity Fourier modes with $1 \leq k \leq 60$ where excited with random phases and with an energy spectrum of k^{-3} for high k, and peaked at $k = 3$. This "noise" was such that added about 10% of the energy already present due to the idealized shear flow. In fact, a lower noise level would be adequate to trigger the nonlinear dynamics, but a larger level was used for reasons that will be further explored in Section 5. Thus, we have initially $E_k = 0.5$, $\Omega = 5.738644$ (Enstrophy), $P = 0.1866146E + 04$ (Palinstrophy) and $Q = 0.2375568E + 07$ (Q-enstrophy). Proceeding from this initial condition, the subsequent dynamics gives rise to a familiar set of phenomena associated with the two dimensional shear layer, including vortex layer breakup, vortex rollup and coalescence of like-signed vortices. These will be described for the spectral and LBE runs in the next section.

Operationally, the spectral run is a standard type, familiar in turbu-

lence computations, using an Orszag-Patterson implementation of a Fourier Galerkin scheme. We use a periodic box of side 2π and resolution 256^2, which allows to use a Reynolds number $R = 10,000$, and still resolve the dissipation wavenumber. This method solves the equation for the vorticity $\omega = (\nabla \times \mathbf{v})_z$,

$$\partial\omega/\partial t + \mathbf{u} \cdot \nabla\omega = \nu\nabla^2\omega, \tag{7}$$

using a fast transform evaluation of the nonlinear couplings, along with appropriate procedures for removal of aliasing errors. The Reynolds number for the longest wavelength is $\sim 1/\nu$. Time integration is a second order explicit scheme, using fixed time steps of $\Delta t = 1/1024$. The characteristic time scale for the motion of the large scale eddies is estimated from the energy as $T_{SP} = L/\sqrt{2E}$, for a characteristic length scale L. In view of the slow decay of the energy, we estimate T_{SP} using the initial value of the energy, and the unit length associated with the longest wavelength in the periodic box. Thus, in simulation units of time, the large scale eddy turnover time is $T_{SP} \approx 1$. The system is evolved up to simulation time $t = 119$. The spectral simulation begins with the specified Fourier coefficients that generate the initial data, and the complete set of vorticity Fourier coefficients are stored at later times for subsequent comparison with the LBE results.

For the LBE run, we obtain the initial fields from the relation $\mathbf{v} = \nabla\psi \times \mathbf{z}$, where $\mathbf{z}$ is the unit vector in the z direction, taking the stream function ψ from the solution to $\nabla^2\psi = -\omega$, which is algebraically solved in Fourier space using the appropriate value of ω from the spectral run. The initial density is set to a constant. These fields are then used to initialize the distribution function f to its equilibrium value, for these specified fields, using Eq.(2). After this initialization procedure, the LBE system is evolved by subjecting it to the sequence of streaming and collisions alluded to above.

For the LBE simulation we used a 512^2 box. For $2D$ hydrodynamics turbulence an estimation of the dissipation length scale L_d can be obtained with $L_d/L_0 \sim R^{-1/2}$, where L_0 is the energy-containing length. For $L_0 = 512$ we get $L_d \sim 5$, that is, the viscous dissipation mechanisms are effectively occurring in a scale of the order of five cells. A smaller run of 256^2 size was carried out; however effects typical of lack of resolution of the dissipation length were observed for this computation. Having chosen the appropriate lattice size and $\sqrt{\langle u^2 \rangle} = 0.04$ for the LBE system, the relaxation parameter τ is then fixed to give the proper viscosity value to obtain $R = 10,000$, using the expression $\nu = (2\tau - 1)/6$ (in lattice units). In particular to achieve an

LBE Reynolds number, i.e $R = UL/\nu = 10,000$, we use

$$R = 0.04 \times \frac{512}{2\pi} \times \frac{1}{(2\tau - 1)/6} \tag{8}$$

arriving as $R = 10,000$ when $\tau = 0.500977848$. Although the physical Reynolds numbers will be time dependent, scaling with the characteristic fluctuating fluid velocities, we expect that these LBE parameters produce Reynolds numbers in the two types of runs that are within 10% in value.

To be able to compare LBE and spectral runs we also have to relate the time units, from lattice convection units (i.e., time needed to propagate microscopic information from cell to cell) to large scale eddy turnover time. That conversion is done in the following way. Using characteristic lengths and velocities for both spectral and LBE systems, we can find a relationship between the characteristic times for the schemes. Thus, using $L_{LBE} = 512$, $L_{SP} = 2\pi$, $U_{LBE} = \sqrt{\langle u^2 \rangle} = 0.04$, $U_{SP} = 1$, we can get an expression connecting the typical time for evolution of both systems,

$$\frac{T_{LBE}}{T_{SP}} = \frac{L_{LBE}/L_{SP}}{U_{LBE}/U_{SP}} = \frac{512}{2\pi}\frac{1}{0.04}, \tag{9}$$

thus

$$T_{LBE} = 2037.12 \;\; T_{SP} \tag{10}$$

so we need about 2037 LBE time steps to complete one spectral characteristic time.

Global features of the evolution are calculated dynamically for the LBE computation every 200 LBE-time steps (i.e. about 1/10 eddy turnover time). The velocity field is scaled to the units used for the spectral simulation and is Fourier transformed. The vorticity $\hat{\omega}(k) = i(\mathbf{k} \times \hat{\mathbf{v}}(\mathbf{k}))_z$ is then evaluated, from which the energy, enstrophy, palinstrophy, q-enstrophy and mean square stream function are computed. This scheme assumes that the fluid is incompressible, i.e., for these global diagnostics, and for energy wavenumber spectra, the (small) admixture of nonvortical velocity fluctuations associated with the compressible LBE dynamics, is ignored. In Section 6 we will further discuss the validity of this approximation.

4 Comparison of Spectral and LBE results

In spite of the organized large scale appearance our initial data - a periodic shear layer perturbed by random fluctuations, the evolution of the system in time is quite typical of 2D incompressible hydrodynamics. Thus, the time histories of global quantities, illustrated in Fig. 1, are familiar in appearance and interpretation. Fig. 1 shows the evolution of the energy $E = \sum_{\mathbf{k}} |\omega(\mathbf{k})|^2/k^2$, the enstrophy $\Omega = \sum_{\mathbf{k}} |\omega(\mathbf{k})|^2$, the Palinstrophy $P = \sum_{\mathbf{k}} k^2|\omega(\mathbf{k})|^2$, and (lacking a better nomenclature) the "Q-enstrophy" $Q = \sum_{\mathbf{k}} k^4|\omega(\mathbf{k})|^2$. (The sums are over the independent wavevectors $\mathbf{k}$.)

E and Ω are inviscid invariants, and therefore are monotonically decreasing in these dissipative simulations. On the other hand P and Q can be amplified as well as dissipated, and are not monotonic. One can also prove that Ω/E decreases in time[19]. This is associated with the tendency for 2D Navier Stokes flow to engage in "selective decay", wherein the turbulence drives the spectrum towards its geometrically determined extremal state $\Omega = K_{min}^2 E$, where K_{min} is the lowest allowed value of wavenumber, in a time short compared with the decay of the flow due to viscosity. The perspective provided by Fig. 1 is consistent with prior results in showing that selective decay is at least a qualitatively useful picture of 2D turbulence. For example, focusing on Fig. 1a) and 1b), one sees that E decays quite slowly compared with Ω, allowing the conclusion to be drawn that the energy is "back-transferred" in k, where dissipation is slow. On the other hand, the flow tends to produce additional amounts of P (see Fig. 1c), an effect that accelerates the dissipation of Ω, since $\dot{\Omega} = -2\nu P$. In addition Q is also amplified early in the run (Fig. 1d), and is dissipated at later times along with E, Ω and P. This complex process of spectral transfer, involving both direct transfer to higher k, and backtransfer to lower k is familiar in 2D flows [20], and is a consequence of a very large number of nonlinear couplings each involving triads of wave vectors. These couplings, as well as their symmetries that give rise to inviscid conservation of E and Ω, are accurately simulated by the spectral method simulation technique. What is new in the panels of Fig. 1 is evidence that the LBE method tracks the spectral method closely with respect to evolution of E, Ω, P and Q. Therefore, even though the LBE method does not involve a wavevector representation, or even the vorticity, in any direct way, it evidently provides a representation of the Navier Stokes dynamics that is accurate enough to preserve the subtleties of 2D nonlinear spectral transfer.

Each of the quantities E, Ω, P and Q, provides a measure of the distribution of vorticity over wavenumber, and those with higher powers of k weight the short wavelengths more heavily. A careful inspection of Fig. 1a-d shows that the quantities that emphasize the lower k part of the spectrum are most similar in the LBE and spectral runs. Evidently, the departures of the LBE from the incompressible spectral method are greatest at the higher wavenumbers. Nevertheless, even fine features of the spectral method evolution of P and Q are also seen in the LBE curves for the same quantities.

Wavenumber spectra of the energy are compared in Fig. 2, for the spectral and LBE results, at times $t = 0, 5, 49$ and 80 (in simulation time units, i.e., eddy turnover times computed in terms of the initial data). Fig. 2a) shows, for the two runs, the initial spectra, which are identical by construction. Local peaks at the lower wavenumbers are associated with the initial shear layers, while the higher k powerlaw is due to the "noise" perturbation. By $t = 5$, a substantial amount of spectral evolution has occurred in both LBE and spectral runs, but, as is shown in Fig. 2b), the energy spectra for the two cases have remained extremely close. The most noticeable departures are at the highest values of k, as expected from the discussion in the previous paragraphs. Very similar energy spectra are also seen at much later time, as is illustrated in Fig. 2c) and 2d) at times $t = 49$ and $t = 80$. In these latter two comparison plots one can see clearly that significant amounts of back transferred energy persists in the longer scales at these late times, and that this effect is accurately portrayed in the LBE run.

Perhaps the most striking verification of the accuracy of the LBE run is found in the direct comparison of contour plots of the LBE vorticity with the spectral method vorticity at the same physical times. In Fig. 3 we show pairs of vorticity contour plots at four times. While the times are given in simulation times, it should be noted that the equivalent LBE time was computed from the calibration discussed in the previous section. The early time state, at $t = 1$, is seen in Fig. 3a), which shows, in both the spectral and LBE cases, that the initial shear layers have begun the familiar process of vortex roll-up. The vorticity distribution is extremely similar in the two cases. By a later time ($t = 5$, in Fig. 3b) the roll-up has progressed and has produced individual vorticity concentrations. These subsequently convect in the flow due to all the vortices, and mergers occur between like-signed vortices due to "vortex collisions". Once again, the plots from the two methods show great similarity, even down to detailed structures near regions of like signed vortex interactions. Note that the same values of

vorticity contours are used in performing the comparisons. A distinctive vortex collision is captured by both methods at time $t = 17$, shown in Fig. 3c.

At later times, all the positive and negative vortex concentrations have separately merged into a single pair of vortices. Fig. 3d shows this state, computed in both the spectral and LBE runs, at $t = 80$. It is clear that the LBE method has succeeded in reproducing many of the important dynamical features obtained by the spectral method, which has been the standard method for turbulence. These features include the evolution of bulk quantities, the form and evolution of the wavenumber spectra, and the detailed features of vorticity contours, including vortex rollup and subsequent mergers of like signed vortices. We now turn to some more subtle features of the flow, which appear also to be well represented by the LBE method.

5 Relaxation to "sinh-Poisson" most probable state

An interesting by-product of the decaying turbulence computation just described concerns the extent to which the two-vortex quasi-steady final state has vortex shapes which coincide with those recently seen at the end of a study of decaying two-dimensional turbulence reported elsewhere. A slight digression is required before it is possible to display the relaxation products of the turbulent computation in a way that will make this connection clear. It has long been realized that in decaying 2D Navier-Stokes flow, enstrophy or mean-square vorticity decayed rapidly compared to energy or mean-square velocity, for reasons that are well known. The separation of the time scales increases with Reynolds number, and had led to a conjecture that the relaxed state of decaying 2D Navier Stokes turbulence would be one in which the enstrophy was minimized relative to the remaining energy[19, 21, 22, 23]. In such a state, the only excitations left in the energy spectrum would be those in the longest wavelengths allowed by the boundary conditions. Qualitatively, such a "selectively decayed" state would resemble, for example, the states shown in the two panels of Fig. 3d. Some time ago[24, 25], a highly-resolved (512x512), high Reynolds number (14,286, based on the largest allowed wavelength), and long-time (about 400 large-scale eddy turnover times) 2D Navier Stokes spectral-method computation was carried out, in an effort to test the above-described "selective decay" hypothesis. In broad

outline, the tests confirmed the hypothesis, but examined closely, departed from it. In particular, a scatter plot of computed pointwise vorticity vs. stream function revealed not a linear proportionality between the two, as the selective decay hypothesis would suggest, but rather a hyperbolic-sinusoidal one, in which the observed connection[26] was that the late-time vorticity and stream function were related by

$$c\omega = \sinh(|\beta|\psi) \tag{11}$$

where c and β are constants. The result was surprising, to the extent that it had been predicted two decades ago[27, 28] from a mean-field theory of most probable states, not for a viscous Navier-Stokes continuum, but rather for a large number of ideal, discrete, parallel line vortices. The subject had developed, with both analytical and numerical solutions of the "sinh-Poisson" partial differential equation[27, 28] that had been derived to describe the most-probable, or maximum-entropy, states, and a good bibliography is given by Smith[29]. A reformulation of the maximum-entropy theory had been given in the context of magnetohydrodynamics[30, 31], and a further development of the foundation of the Navier-Stokes basis for it will be given elsewhere[32]. Our intent in this Section is simply to point out that even this somewhat unexpected and perhaps exotic hyperbolic-sine connection between stream function and vorticity has been reproduced accurately in the present LBE computation. In Fig. 4, we graph two correlation functions vs. time, with the broken line referring to the spectral method computation and the solid line to the LBE computation. Shown in Fig. 4 are correlations between vorticity and stream function (lower curves) and between vorticity and the hyperbolic sine of β times the stream function (upper curves), where the constant β is determined from a least-squares fit to the computed data. For any two functions f(x,y) and g(x,y), the correlation C(f,g) is defined by

$$C(f,g) \equiv \frac{\langle (f-\langle f\rangle)(g-\langle g\rangle)\rangle}{[\langle (f-\langle f\rangle)^2\rangle\langle (g-\langle g\rangle)^2\rangle]^{1/2}} \tag{12}$$

where the angle brackets $\langle\rangle$ denote a spatial average over the entire box. Thus for any two functions which are proportional, C will be equal to unity. The approach to the "sinh-Poisson" prediction is seen not only to be far superior for the computed data, but it will also be noticed that the LBE and spectral method computations again track each other to a remarkable extent. We remark also that there is a problem, as yet unsolved, of extracting the observed statistical mechanical distribution of the LBE variables for

a vortex distribution directly from the LBE dynamics, without the necessity of detouring through the Navier-Stokes approximation. This must at present stand as a challenge for theory; a solution would be highly desirable as a logical link between the microscopic and macroscopic dynamics.

6 Nearly incompressible hydrodynamics in the LBE scheme

In the previous sections, evidence was presented that the LBE method reproduces many of the essential dynamical features of the *incompressible* Navier Stokes equations, as computed by a spectral method code based upon the vorticity equation. In particular, we found that the solutions appear to be quantitatively similar to one another. What differences there are between the spectral and LBE results appear to be most pronounced at the higher wavenumbers. It is tempting to assign these discrepancies to "error" in the LBE formulation, and conclude that the methods correspond well, for most of the diagnostics of interest, at times of up to tens, or perhaps a hundred or more, characteristic nonlinear times.

However, there remains the possibility that the LBE scheme, is, in some sense, more accurate than this would suggest. We refer here to the possibility that the LBE scheme, in effect is solving a *compressible* set of fluid equations, and therefore, would be expected to approximate solutions of the incompressible equations only in an appropriately defined limiting sense. In fact, the compressible Navier Stokes equation itself also possesses this property. For suitably chosen initial data, and for small Mach numbers, the solutions of the compressible fluid equations are expected [33] to approximate the solutions to the incompressible equations for at least some finite time interval. Simulations [34] of the compressible equations of 2D hydrodynamics (with a polytropic equation of state) have also led to the suggestion that finite Reynolds number extends the realm of this expectation, so that in some cases the "nearly incompressible" nature of a decaying flow may persist permanently. Since the LBE method is intrinsically compressible, we can reasonably expect that it, too, will admit a range of parameters and time in which its solutions approach the desired incompressible solutions. This, indeed, is what we have seen in the previous section. However, there is also the prospect that some of the *departures* of the LBE solutions from the spectral method incompressible solutions might be attributable to the

slight effects of compressibility. In that case, at least some fraction of the differences between the spectral and LBE solutions might not be errors of numerical origin, but rather physical effects that lie outside the realm of the incompressible equations. We briefly explore this possibility here, by examining whether the LBE results are consistent with the expectations of nearly incompressible fluid theory[33].

Equation (5), the incompressible equation for the velocity field, assumes that the leading order velocity field, say, $\mathbf{v}_0$, is divergenceless, $\nabla \cdot \mathbf{v}_0 = 0$ and the leading order density is constant, say $\rho = \rho_0 =$ const.. Again assuming that this limit to incompressibility is obtained, we find that the incompressible pressure p^∞ appearing in (5), must satisfy

$$\nabla^2 p^\infty = -\rho_0 \nabla \cdot (\mathbf{v}_0 \cdot \nabla \mathbf{v}_0) \tag{13}$$

which is a consequence of the time independence of $\nabla \cdot \mathbf{v}_0 = 0$. On the other hand, prior to the limit to incompressibility, the LBE system is found to obey the compressible equations

$$\rho[\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}] = -\nabla p + \mathbf{D}, \tag{14}$$

which, along with the continuity equation (6) and the equation of state $p = C_s^2 \rho$ completes the specification of the long wavelength, low frequency LBE dynamics. The term $\mathbf{D}$ on the right side of Eq. (14) represents the viscous dissipation terms for the *compressible* fluid limit of the LBE method. Since neither the form nor the effects of dissipative terms are central to the description of near-incompressibility that we examine here, we neglect $\mathbf{D}$ in the following discussion.

Let us define the Mach number of the flow as $M \equiv \delta v / C_s$, with δv the rms value of $\mathbf{v}$. When $M << 1$ we expect there to be conditions in which a decaying flow will remain nearly incompressible. Klainerman and Majda[33] have shown that the additional required conditions are that the initial data satisfy $\langle |\nabla \cdot \mathbf{v}|^2 \rangle^{1/2} = O(M)$ and $\delta\rho = \langle (\rho - \rho_0)^2 \rangle^{1/2} = O(M^2)$ where $\langle ... \rangle$ denotes a volume average. In the LBE run discussed above, the initial $\delta v = 0.04$, and $C_s = 1/\sqrt{3}$, so the initial $M = 0.069$. In addition, $\rho = \rho_0$ is uniform in the initial data. As for the velocity field, it is computed for the LBE initially in terms of the real space values obtained from the spectral method initial data. Thus, except for possibly errors due to the finite LBE grid, it satisfies $\nabla \cdot \mathbf{v} = 0$ initially. Consequently, the conditions for the approach of the compressible equations to the solutions of the incompressible equations appear to be well fulfilled.

In this circumstance, we expect that, for a finite time, the density should remain ordered as $\rho = \rho_0 + M^2(\rho^\infty + \rho') + O(M^3)$, while the pressure (in convection speed units) should satisfy $p = M^{-2}(p_0 + M^2(p^\infty + p') + O(M^3))$. Here, p^∞ is the incompressible pressure, satisfying the Poisson equation (13). There is also an additional pressure fluctuation p', at the same order as the incompressible pressure, but associated with acoustic waves, and decoupled from the incompressible equation of motion. The leading order density fluctuation is $\delta\rho \approx M^2(\rho^\infty + \rho') = \delta\rho^\infty + \delta\rho'$, where ρ' is also associated with acoustic waves. In addition to the Poisson equation, the incompressible pressure satisfies the relation $p^\infty + p' \approx C_s^2 \delta\rho$. In order for the incompressible dynamical equation to lack acoustic time scale variations, we must apportion the leading order density fluctuations so that $p^\infty = C_s^2 \delta\rho^\infty = M^{-2}\delta\rho^\infty$, the latter equality making use of the convection speed units. Considering also the velocity field, we note that, in a Fourier decomposition, we can readily divide the velocity field as $\mathbf{v} = \mathbf{v}_L + \mathbf{v}_\perp$ where the longitudinal velocity $\mathbf{v}_L$ has $\nabla \cdot \mathbf{v}_L \neq 0$ but $\nabla \times \mathbf{v}_L = 0$, while the transverse velocity $\mathbf{v}_\perp$ satisfies $\nabla \cdot \mathbf{v}_\perp = 0$ but $\nabla \times \mathbf{v}_\perp \neq 0$. Then, for maintaining near-incompressibility we require [33] that the solutions remain ordered so that $\mathbf{v}_L = O(M)$ for (incompressible) convection speed units in which $\mathbf{v}_\perp = O(1)$.

The degree to which these expectations of nearly incompressible fluid theory are seen in the LBE solution can be examined by analysis of the LBE velocity and density fields. The results are illustrated in the panels of Fig. 5. The LBE velocity field is Fourier transformed and decomposed into transverse and longitudinal components by projections relative to wave vector $\mathbf{k}$. Then, the rms transverse velocity $U_\perp = \sqrt{\langle|\mathbf{v}_\perp|^2\rangle}$ and the rms longitudinal velocity $U_L = \sqrt{\langle|\mathbf{v}_L|^2\rangle}$ are computed. The relative magnitudes of $U_\perp$ and U_L during the simulation are shown in Figs. 5a and 5b. In Fig. 5a, the time history of $U_\perp/C_s$ is shown. The value decreases slightly from its initial value, which is very close to the value of the Mach number $M = 0.069$ computed from the entire velocity field. Thus, we expect that the longitudinal velocity is small, and this is verified in Fig. 5b, which shows $U_L/U_\perp$ as a function of time. The latter ratio meanders about a value near 0.010. Consequently, in convection speed units, it is clear that the condition $U_L = O(M)$, required for the approach to incompressibility, is well satisfied.

The density fluctuations may be decomposed as well, in accordance with nearly incompressible theory. First, we simply evaluate the total density fluctuation $\delta\rho$, and compare its value to the mean density and the Mach number as time progresses. This is shown in Fig. 5c as the solid trace, $\delta\rho/\rho_0$

vs. time. We see that the magnitude of the root mean square total density fluctuation is comparable to the expected value, of order $M^2 \approx 0.0048$. Next, we decompose the density into the part associated with the underlying incompressible flow, and the part associated with acoustic activity. Using only the transverse velocity field, we numerically solve Eq. (13) for the incompressible pressure p^∞. The incompressible density fluctuation is computed as $\delta\rho^\infty = M^2 p^\infty$. The root mean square value of $\delta\rho^\infty$ is plotted also in Fig. 5c, normalized to the mean density. Again the result is clearly $O(M^2)$. Finally, we compute the density fluctuation associated with leading order acoustic effects through $\delta\rho' = \delta\rho - \delta\rho^\infty$ at each point in space. Computing the root mean square $\delta\rho'$ provides a measure of the degree of acoustic activity. This is illustrated as well in Fig. 5c, showing that this component of the density fluctuation also remains of $O(M^2)$, again in accordance with the expectations of nearly incompressible theory.

7 Discussion and Conclusions

In the above sections we have presented a detailed comparison of solutions to the two dimensional Navier Stokes equations obtained from a Lattice Boltzmann method and from a more traditional spectral method. The flow problem considered was a familiar shear layer initial value problem, in periodic boundaries and prepared initially with a low level of random noise. We find that the LBE method has provided a solution that is "accurate" in the sense that time histories of global quantities, wavenumber spectra, and vorticity contour plots, are very closely similar to those obtained from the spectral method. While the comparison is best at early times, the solutions remain extremely close to one another for at least several eddy turnover times, and in some ways remain close for times up to a hundred turnover times. In particular, details of the wavenumber spectra at high wavenumbers are reproduced, as well as the detailed structure of vortex distributions seen in the contour plots. In addition, the LBE scheme faithfully reproduces the recently reported long time tendency for the stream function to approach a "sinh-Poisson" state that emerges from a maximum entropy argument. We have also explored the possibility that the LBE solution, to the extent that it departs from a pure solution of the incompressible equations, is remaining in the mathematically delineated regime of "nearly incompressible flow". This indeed appears to be the case, although a more complete verification would require comparison with a fully compressible

spectral algorithm, a refinement we have not as yet undertaken.

It remains to discuss the accuracy of the LBE scheme in a quantitative way. To do so we have computed several kinds of normalized differences between the results of the two runs, which are interpreted (for the most part) as errors in the LBE method. The normalized errors in the bulk quantities, energy, mean square stream function and enstrophy, are computed, for example, as $|E_{SP} - E_{LBE}|/E_{SP}$, and shown in Table I. (The suffixes SP and LBE refer to ϕ computed from the spectral or LBE schemes, respectively.) The normalized total rms error, defined for the spatially dependent variable ϕ as

$$\varepsilon(\phi) = \left(\frac{\langle (\phi_{SP} - \phi_{LBE})^2 \rangle}{\langle \phi_{SP}^2 \rangle} \right)^{\frac{1}{2}} \tag{15}$$

This rms normalized error has been computed as a function of time for ϕ taken as ω, $\mathbf{v}_\perp$ or ψ. In addition we have computed the kurtosis $K(\phi) = \langle \phi^4 \rangle / \langle \phi^2 \rangle^2$ for ϕ taken as ω, $\mathbf{v}_\perp$ or ψ. Error in the kurtosis is conveniently expressed as $\Delta K/K = |K_{SP} - K_{LBE}|/K_{SP}$, where the suffixes have the same meaning as above. In table I we give the values of these normalized errors at spectral method times $t = 1$, 10, 50 and 100.

		TIME			
Error		1	10	50	100
ε	ψ	0.00742	0.04867	0.35675	1.12567
	v	0.02136	0.13685	0.38283	1.21901
	ω	0.12751	0.53799	0.65278	1.37693
$\Delta K/K$	ψ	0.00097	0.00623	0.00984	0.00161
	v	0.00297	0.01966	0.03168	0.08017
	ω	0.00237	0.01245	0.05960	0.05869
$\Delta\Phi/\Phi$	$\langle\psi^2\rangle$	0.00043	0.00957	0.01676	0.01843
	E	0.00081	0.00075	0.00045	0.00035
	Ω	0.01252	0.01053	0.01500	0.00689

It is immediately apparent that, at any fixed time, and for most categories of error analysis, the error in ψ is smallest, and the error in ω is largest. In keeping with our previous discussion of the comparison of the

spectra, this is associated with the fact that the fractional error in the higher wavenumber excitations are greater than that of the lower wavenumbers.

It is also apparent that the errors in the kurtosis are much smaller than the total rms errors, for a given field. In addition, the error in the bulk energy is less than the error in the kurtosis of the velocity. In fact, the kurtosis errors remain small compared to the total rms error, especially for the vorticity. The reasons for this appear to be that the structures, and the distribution of structures in the LBE run remain quite close to their spectral method counterparts. However, the exact positions of the structures become different in the LBE case, relative to the spectral case. This disparity appears first in the high wavenumber structures, and later on in the large scale structures, so that by $t = 80$ (see Fig. 3d) the large vortices that remain are *not* at the same locations in the two runs. Nevertheless the spectra remain very close (see Fig. 2). As with the spectra, the kurtosis calculation is not sensitive to the position of structures, but only to their magnitude and shape, and, in a statistical sense, to the distribution of shapes. Evidently, the distribution of excitations, in both wavenumber and real space, remains relatively close for the two methods. The largest error appears to be in the *position* of the vorticity structures, and the large increase in the error at later times is associated with the progressive drift in position of the LBE relative to the spectral results.

The origin of the drift in vortex positions, while bulk quantities, shapes and spectra remain fairly accurate, can be attributed to several possible causes. First of all, to compare the methods, we needed to reconcile the LBE timescale with the spectral (fluid) timescale. This was accomplished (see Section 3) in the present study by computing a conversion factor at $t = 0$ giving the ratio of the characteristic time units, involving the rms fluid velocity fluctuation. The latter quantity changes in time, but this change would not produce a difference in the results of the two methods if the fluid kinetic energy and the enstrophy remained exactly equal for the two cases. However, there is a small difference in the energies and enstrophies (see Fig. 1 and Table I), and this causes a slight inaccuracy in the times at which we compared the results. As these “clocks” drift apart, so do the positions of the vortices at the times at which we compare them. This part of the positional drift may be operational in our study, rather than intrinsic to the differences in the numerical methods, and could, in principle, be reduced by a more sophisticated, and more difficult, analysis of the data. A second cause of the positional drift, is closely related to the first, but is

of physical origin. Specifically, we have argued in Sec. 6, that some of the small departures from incompressible flow in the LBE method may be a real physical effect, that of nearly incompressible flow, which the LBE represents reasonably well, but which is absent in the purely incompressible spectral run. The effects of the small amount of compressible flow may include differences in the decay of energy in the two cases, as well as differences in the position of vortices, even at the same physical time into each run. In this perspective, the positional drift, as well as other differences in the results of the two methods, may be attributable to compressive effects, and not numerical error. We note that both of these possible sources of differences in the methods, are expected to have greater influence on the total rms error than on the bulk errors or the kurtosis error. This is because each of them induce small changes in the effective times of a comparison. During this small time increment the position of vortices vary rapidly compared to changes in the spectra, or compared to changes in their shapes (except possibly at times of vortex collisions). The total rms error is extremely sensitive to exact positions of all structures in the simulation domain, even if the structures are otherwise accurately represented.

We are led to the conclusion that the LBE scheme has matured to the point that it offers an alternative method for solving incompressible flow problems with reasonably high accuracy. In particular, the above error analysis suggests that the LBE approach gives relatively good results for bulk quantities such as energy, for wavenumber spectra and for measures of distributions such as kurtosis. Although contour plots show great similarity in spectral and LBE cases, there is, evidently, a growing drift in relative positions of vortex structures in the two cases. However, for turbulence calculations, the importance of exact positions of the vortices is rarely considered central, while spectra, energy decay rates, and statistics such as kurtosis are of great interest. Moreover, we find some indication that the scheme also offers quantitative information concerning the small effects of compressibility, including "pseudosound" density fluctuations associated with the incompressible flow, and accompanying acoustic waves. As far as efficiency is concerned, we note that, for these resolutions and at the Mach number used, the 256^2 spectral run "costs" about 6 cpu minutes per characteristic time, whereas the 512^2 LBE run "costs" about 8 min per characteristic time on the San Diego Cray YMP. Thus, the LBE is of comparable efficiency, and may fare better than the incompressible spectral code in a parallel implementation. However, one should also note that the timings of a *compressible* spectral code would be expected to be about a fac-

tor of M^{-1} longer to resolve acoustic frequencies. Consequently, if the small compressible effects are required, the LBE may already be more efficient.

The particular LBE model we have used is the product of several refinements to the method. These include corrections to the pressure that enforce a particular (isothermal) equation of state, and the use of a single time relaxation procedure for handling the collisional approach to local equilibrium. Further refinements and extensions are also feasible as well. In particular, the pressure can, in principle, be further modified to include an independent temperature variable, so that a full ideal gas equation of state can be implemented. In addition, the method can be modified [17] to allow for higher Mach number flows, and even transonic flows, to be computed. However, this has not been attempted here, in view of our goal of comparison with an incompressible solution, approached through a low Mach number flow.

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Appendix

In developing the LBE theory it is of interest to understand the relationship the theory has to kinetic theory of ordinary gases, in addition to evaluating the computational method itself. In this respect the LBE method described herein possesses some properties that are unusual from the ideal gas kinetic theory perspective. Specifically, the present model is developed to arrive at a useful computational representation of incompressible flow, evidenced by the emergence of Eq. (5) at lowest order in the Chapman Enskog expansion, and also at leading order in a Mach number expansion. However, particularly in

view of recent efforts [35] to employ related LBE methods to flows that may be strongly compressible, it is important to examine features of the method, such as the viscosity, when compressible effects are included. Although a complete examination of these effects has yet to be completed, we have noted the following disparity between the simple STRA LBE method and ordinary gas kinetic theory.

In the kinetic theory of simple gases, the kinematic viscosity is expected to be dependent upon density, approximately as $\nu = \mu/\rho$, where the molecular viscosity μ is approximately independent of density[36, 37]. This scaling emerges because one finds that $\mu \propto \rho v_{th} \lambda$, where v_{th} is a thermal speed (roughly analogous to the LBE lattice streaming speed) and λ is a collisional mean free path, related to a collision time τ_c by $\lambda = v_{th}\tau_c$. In spite of what appears as an explicit linear dependence of μ upon ρ, it is a familiar result that the molecular viscosity is nearly density independent because λ (or, equivalently τ_c) scales as $\propto 1/\rho$. More precisely, on the basis of kinetic theory, molecular viscosity is independent of density for a fixed temperature, a fact originally noted by Maxwell, and born out in standard kinetic theory calculations (e.g.,[37]). However, when such calculations are carried out with a single time relaxation approximation to the collision operator (with relaxation time τ_c), the correct scaling is obtained only by associating with τ_c an inverse proportionality with density.

The STRA LBE method used here and elsewhere[10, 11, 12, 35] differs from the ordinary gas kinetic theory result in that the relaxation time has typically been chosen as a density independent constant. Consequently, there are features of the LBE viscosity that differ from the ordinary gas situation. Most importantly, the molecular viscosity μ is *not* independent of density, essentially because the combination $\rho\tau$ still depends on density. The molecular viscosity cannot be immediately "pulled through" spatial derivatives, divided by ρ, and renamed as the kinematic viscosity ν. Instead there are also new terms that appear, all of which involve $\nabla\rho$. This changes the form of the compressible dissipation terms ($\mathbf{D}$ in Eq. 14) to something other than the precise form expected for a compressible ideal gas. However, these additional terms, according to the nearly incompressible flow theory reviewed in Sec. 6, involve two more factors of Mach number than do the "usual" terms in the viscosity. Thus, the added effects do not directly or significantly influence the incompressible flow component of the LBE in the nearly incompressible regime.

These differences reflect the fact that in LBE theory, in contrast to ordi-

nary gases (as well as cellular automata [1, 2, 3]), the collisional mean free part is not determined by actual collisions that occur in the dynamics. Instead the "collision rate" is determined by the selected relaxation parameter that controls the rate of approach to local equilibrium. This parameter τ is externally controlled and is arbitrary within the bounds set by stability conditions for the LBE dynamical equation. Accordingly, the constant STRA collision operator is adequate, and perhaps also an efficient way, to compute incompressible or nearly incompressible flow with an LBE scheme. However, an improvement may be desirable for LBE schemes that are designed for higher Mach number flows that admit more effects of compressibility[35]. In particular, the STRA model can be modified by choosing $\tau = \tau_0\rho_0/\rho$ with τ_0 a constant time scale, ρ_0 the mean density, and ρ the local value of density. This modification is expected to bring a compressible LBE scheme into closer agreement with the kinetic physics of an ideal gas, particularly with regard to the structure and value of the viscous transport coefficients.

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8 Figure Captions

Fig. 1 Time history of a) energy, b) enstrophy, c) palinstrophy, and d) the next higher order moment, q-enstrophy ($\sim k^4\omega(k)^2$). Continuous line corresponds to the LBE simulation. Departures are noticeable for the higher moments only.

Fig. 2 Wavenumber energy spectra for times a) 0, b) 5, c) 49 and d) 80, for both spectral and LBE simulations. Continuous line corresponds to the LBE simulation. The spectra for $t = 0$ are identical for both runs by construction.

Fig. 3 Isovorticity contour plots for times a) 1, b) 5, c) 17, and d) 80. Dashed lines correspond to negative values of vorticity. The values for the contours are the same for all cases. Strikingly similar features can be found for the LBE simulation as compared with the spectral simulation.

Fig. 4 Correlation between ω and ψ, and between ω and $\sinh(|\beta|\psi)$ as a function of time. Continuous line corresponds to the LBE simulation.

Fig. 5. Near incompressibility of the LBE run. a) time history of the rms transverse velocity normalized by the sound speed $U_\perp/C_s$. This quantity remains approximately constant, and equal to the initial Mach number $M = 0.069$. b) $U_L/U_\perp$ as a function of time, where U_L is the rms longitudinal velocity. This ratio is clearly bounded by M, as required for approaching incompressibility. c) Density fluctuations divided by ρ_0 as a function of time for the LBE simulation. ρ, ρ^∞ and $\rho\prime$ correspond to the total density, the "incompressible" density, and density fluctuations associated with acoustic waves, respectively (see text). All fluctuations are $O(M^2)$ ($M^2 = 0.0048$), consistent with nearly incompressible theory.

9 Table Captions

Table. 1. Normalized differences between the spectral run and the LBE run for various quantities for $t = 1$, 10, 50 and 100. ε is the total rms error, whereas $\Delta\Phi/\Phi = |\Phi_{SP} - \Phi_{LBE}|/\Phi_{SP}$. Large differences in ε are due mainly to a drift in vortex positions. Differences are significantly reduced for the two lower sections of the table that show errors in quantities that are independent of the exact distribution of vorticity but are, instead, sensitive to the shape of the vortices.